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ON SOME UNIVERSAL SUMS OF GENERALIZED POLYGONAL NUMBERS

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Abstract. For $m = 3, 4, \ldots$ those $p_m(x) = (m-2)x(x-1)/2 + x$ with $x \in \mathbb{Z}$ are called generalized m-gonal numbers. Recently the second author studied for what values of positive integers a, b, c the sum $ap_5 + bp_5 + cp_5$ is universal over $\mathbb{Z}$ (i.e., any $n \in \mathbb{N} = \{0, 1, 2, \ldots\}$ has the form $ap_5(x) + bp_5(y) + cp_5(z)$ with $x, y, z \in \mathbb{Z}$). In this paper we proved that $p_5 + bp_5 + 3p_5$ $(b = 1, 2, 3, 4, 9)$ and $p_5 + 2p_5 + 6p_5$ are universal over $\mathbb{Z}$; this partially confirms Sun's conjecture on $ap_5 + bp_5 + cp_5$. Sun also conjectured that any $n \in \mathbb{N}$ can be written as $p_3(x) + p_5(y) + p_{11}(z)$ and $3p_3(x) + p_5(y) + p_7(z)$ with $x, y, z \in \mathbb{N}$; in contrast we show that $p_3 + p_5 + p_{11}$ and $3p_3 + p_5 + p_7$ are universal over $\mathbb{Z}$.

1. Introduction

For $m = 3, 4, \ldots$ we set

$$p_m(x) = (m-2)\frac{x(x-1)}{2} + x. \tag{1.1}$$

Those $p_m(n)$ with $n \in \mathbb{N} = \{0, 1, 2, \ldots\}$ are the well-known m-gonal numbers (or polygonal numbers of order m). We call those $p_m(x)$ with $x \in \mathbb{Z}$ generalized m-gonal numbers. Note that (generalized) 3-gonal numbers are triangular numbers and (generalized) 4-gonal numbers are squares of integers.

In 1638 Fermat asserted that each $n \in \mathbb{N}$ can be written as the sum of m polygonal numbers of order m. This was proved by Lagrange, Gauss and Cauchy in the cases $m = 4$, $m = 3$ and $m \geqslant 5$ respectively (see Chapter1 of [N96, pp. 3-34]) The generalized pentagonal numbers are related to the partition function as discovered by Euler (see, e.g., Berndt [B, p. 12]).

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For $a, b, c \in \mathbb{Z}^+ = \{1, 2, 3, \dots\}$ and $i, j, k \in \{3, 4, \dots\}$, Sun [S09] called the sum $ap_i + bp_j + cp_k$ universal over $\mathbb{N}$ (resp., over $\mathbb{Z}$) if for any $n \in \mathbb{N}$ the equation $n = ap_i(x) + bp_j(y) + cp_k(z)$ has solutions over $\mathbb{N}$ (resp., over $\mathbb{Z}$). In 1862 Liouville (cf. [D99, p. 23]) determined all those universal $ap_3 + bp_3 + cp_3$. The second author [S07] initiated the determination of those universal sums $ap_i + bp_j + cp_k$ with $\{i, j, k\} = \{3, 4\}$, and this project was completed via [S07], [GPS] and [OS].

It is known that generalized hexagonal numbers are identical with triangular numbers (cf. [G94] or [S09, (1.3)]).

The second author recently established the following result.

Theorem 1.0 (Sun [S09]). *Suppose that $ap_k + bp_k + cp_k$ is universal over $\mathbb{Z}$, where $k \in \{4, 5, 7, 8, 9, \dots\}$, $a, b, c \in \mathbb{Z}^+$ and $a \leqslant b \leqslant c$. Then $k = 5$, $a = 1$ and (b, c) is among the following 20 ordered pairs:*

$$\begin{gathered}(1,k)\ (k\in[1,10]\setminus\{7\}),\\ (2,2),\ (2,3),\ (2,4),\ (2,6),\ (2,8),\\ (3,3),\ (3,4),\ (3,6),\ (3,7),\ (3,8),\ (3,9).\end{gathered}$$

Guy [G94] realized that $p_5 + p_5 + p_5$ is universal over $\mathbb{Z}$, and Sun [S09] proved that the sums

$$p_5+p_5+2p_5,\ p_5+p_5+4p_5,\ p_5+2p_5+2p_5,\ p_5+2p_5+4p_5,\ p_5+p_5+5p_5$$

are universal over $\mathbb{Z}$. So the converse of Theorem 1.0 reduces to the following conjecture of Sun.

Conjecture 1.0 (Sun [S09]). *The sum $p_5 + bp_5 + cp_5$ is universal over $\mathbb{Z}$ if the ordered pair (b, c) is among*

$$\begin{gathered}(1,3),\ (1,6),\ (1,8),\ (1,9),\ (1,10),\ (2,3),\ (2,6),\\ (2,8),\ (3,3),\ (3,4),\ (3,6),\ (3,7),\ (3,8),\ (3,9).\end{gathered}$$

Our first result confirms this conjecture partially.

Theorem 1.1. *For*

$$(b,c) = (1,3), (2,3), (2,6), (3,3), (3,4), (3,9),$$

the sum $p_5 + bp_5 + cp_5$ is universal over $\mathbb{Z}$.

Sun [S09] investigated those universal sums $ap_i + bp_j + cp_k$ over $\mathbb{N}$. By Conjectures 1.1 and 1.2 of Sun [S09], $p_3+p_5+p_{11}$ and $3p_3+p_5+p_7$ should be universal over $\mathbb{N}$. Though we cannot prove this, we are able to show the following result.

Theorem 1.2. *The sums $p_3 + p_5 + p_{11}$ and $3p_3 + p_5 + p_7$ are universal over $\mathbb{Z}$.*

Theorems 1.1 and 1.2 will be proved in Sections 2 and 3 respectively.

2. Proof of Theorem 1.1

Lemma 2.1 (Sun [S09, Lemma 3.2])**.** *Let $w = x^2 + 3y^2 \equiv 4 \pmod 8$ with $x, y \in \mathbb{Z}$. Then there are odd integers u and v such that $w = u^2 + 3v^2$.*

Lemma 2.2. *Let $w = x^2 + 3y^2$ with x, y odd and $3 \nmid x$. Then there are integers u and v relatively prime to 6 such that $w = u^2 + 3v^2$.*

Proof. It suffices to consider the case $3 \mid y$. Without loss of generality, we may assume that $x \not\equiv y \pmod 4$ (otherwise we may use $-y$ instead of y). Thus $(x-y)/2$ and $(x+3y)/2 = (x-y)/2 + 2y$ are odd. Observe that

$$x^2 + 3y^2 = \left(\frac{x+3y}{2}\right)^2 + 3\left(\frac{x-y}{2}\right)^2. \tag{2.1}$$

As $3 \nmid x$ and $3 \mid y$, neither $(x-y)/2$ nor $(x+3y)/2$ is divisible by 3. Therefore $u = (x+3y)/2$ and $v = (x-y)/2$ are relatively prime to 6. This concludes the proof. □

Lemma 2.3 (Jacobi's identity)**.** *We have*

$$3(x^2 + y^2 + z^2) = (x+y+z)^2 + 2\left(\frac{x+y-2z}{2}\right)^2 + 6\left(\frac{x-y}{2}\right)^2. \tag{2.2}$$

We need to introduce some notation. For $a, b, c \in \mathbb{Z}^+ = \{1, 2, 3, \dots\}$, we set

$$E(ax^2 + by^2 + cz^2) = \{n \in \mathbb{N} : \ n \neq ax^2 + by^2 + cz^2 \text{ for any } x, y, z \in \mathbb{Z}\}.$$

Proof of Theorem 1.1. Let $b, c \in \mathbb{Z}^+$. For $n \in \mathbb{N}$ we have

$$\begin{aligned} n = p_5(x) + bp_5(y) + cp_5(z) &= \frac{3x^2 - x}{2} + b\frac{3y^2 - y}{2} + c\frac{3z^2 - z}{2} \\ \iff 24n + b + c + 1 &= (6x-1)^2 + b(6y-1)^2 + c(6z-1)^2. \end{aligned}$$

If $w \in \mathbb{Z}$ is relatively prime to 6, then w or $-w$ is congruent to -1 modulo 6. Thus, $p_5 + bp_5 + cp_5$ is universal over $\mathbb{Z}$ if and only if for any $n \in \mathbb{N}$ the equation $24n + b + c + 1 = x^2 + by^2 + cz^2$ has integral solutions with x, y, z relatively prime to 6.

Below we fix a nonnegative integer n.

(i) By Dickson [D27, Theorem III],

$$E(x^2 + y^2 + 3z^2) = \{9^k(9l+6) : \ k, l \in \mathbb{N}\}. \tag{2.3}$$

So $24n + 5 = u^2 + v^2 + 3w^2$ for some $u, v, w \in \mathbb{Z}$. As $3w^2 \not\equiv 5 \pmod 4$, u or v is odd. Without loss of generality we assume that $2 \nmid u$. Since

$v^2+3w^2 \equiv 5-u^2 \equiv 4 \pmod 8$, by Lemma 2.1 we can rewrite v^2+3w^2 as s^2+3t^2 with s,t odd. Now we have $24n+5=u^2+s^2+3t^2$ with u,s,t odd. By $u^2+s^2 \equiv 5 \equiv 2 \pmod 3$, both u and s are relatively prime to 3. Applying Lemma 2.2 we can express s^2+3t^2 as y^2+3z^2 with y,z relatively prime to 6. Thus $24n+5=u^2+y^2+3z^2$ with u,y,z relatively prime to 6. This proves the universality of $p_5+p_5+3p_5$ over $\mathbb{Z}$.

(ii) By Dickson [D27, Theorem X],

$$E(x^2+2y^2+3z^2)=\{4^k(16l+10):\ k,l\in\mathbb{N}\}. \tag{2.4}$$

So $24n+6=2u^2+v^2+3w^2$ for some $u,v,w\in\mathbb{Z}$. Clearly v and w have the same parity. Thus $4 \mid v^2+3w^2$ and hence $2u^2\equiv 6 \pmod 4$. So u is odd and $v^2+3w^2\equiv 6-2u^2\equiv 4 \pmod 8$. By Lemma 2.1 we can rewrite v^2+3w^2 as s^2+3t^2 with s,t odd. Now we have $24n+6=2u^2+s^2+3t^2$ with u,s,t odd. Note that $s^2+2u^2>0$ and $s^2+2u^2\equiv 0 \pmod 3$. By [S09, Lemma 2.1], we can rewrite s^2+2u^2 as x^2+2y^2 with x,y relatively prime to 3. As $x^2+2y^2=s^2+2u^2\equiv 3 \pmod 8$, both x and y are odd. By Lemma 2.2, $x^2+3t^2=r^2+3z^2$ for some integers $r,z\in\mathbb{Z}$ relatively prime to 6. Thus $24n+6=r^2+2y^2+3z^2$ with r,y,z relatively prime to 6. It follows that $p_5+2p_5+3p_5$ is universal over $\mathbb{Z}$.

(iii) By Dickson [D27, Theorem IV],

$$E(x^2+3y^2+3z^2)=\{9^k(3l+2):\ k,l\in\mathbb{N}\}. \tag{2.5}$$

So $24n+7=u^2+3v^2+3w^2$ for some $u,v,w\in\mathbb{Z}$. Since $u^2\not\equiv 7 \pmod 4$, without loss of generality we assume that $2\nmid w$. As $u^2+3v^2\equiv 7-3w^2\equiv 4 \pmod 8$, by Lemma 2.1 there are odd integers s and t such that $u^2+3v^2=s^2+3t^2$. Thus $24n+7=s^2+3t^2+3w^2$ with s,t,w odd. Clearly s is relatively prime to 6. By Lemma 2.2, $s^2+3t^2=x_0^2+3y^2$ for some integers x_0 and y relatively prime to 6, and $x_0^2+3w^2=x^2+3z^2$ for some integers x and z relatively prime to 6. Therefore $24n+7=x^2+3y^2+3z^2$ with x,y,z relatively prime to 6. This proves the universality of $p_5+3p_5+3p_5$ over $\mathbb{Z}$.

(iv) By [S09, Theorem 1.2(iii)], $24n+8=u^2+v^2+3w^2$ for some $u,v,w\in\mathbb{Z}$ with $2\nmid w$. Clearly $u\not\equiv v \pmod 2$. Without loss of generality, we assume that $u=2r$ with $r\in\mathbb{Z}$. Since $(2r)^2+v^2\equiv 8\equiv 2 \pmod 3$, both r and v are relatively prime to 3. As v and w are odd, $v^2+3w^2\equiv 4 \pmod 8$ and hence r is odd. By Lemma 2.2, we can rewrite v^2+3w^2 as x^2+3y^2 with x and y relatively prime to 6. Note that $24n+8=4r^2+v^2+3w^2=x^2+3y^2+4r^2$ with x,y,r relatively prime to 6. It follows that $p_5+3p_5+4p_5$ is universal over $\mathbb{Z}$.

(v) By (2.3), $24n+13=u^2+v^2+3w^2$ for some $u,v,w\in\mathbb{Z}$. Since $3w^2\not\equiv 13\equiv 1 \pmod 4$, without loss of generality we may assume that u is odd. As $v^2+3w^2\equiv 13-u^2\equiv 4 \pmod 8$, by Lemma 2.1 we can rewrite

v^2+3w^2 as s^2+3t^2 with s and t odd. Thus $24n+13=u^2+s^2+3t^2$ with u,s,t odd. Since $u^2+s^2\equiv 13\equiv 1 \pmod 3$, without loss of generality we may assume that $3\nmid u$ and $s=3r$ with $r\in\mathbb{Z}$. By Lemma 2.2, $u^2+3t^2=x^2+3y_0^2$ for some integers x and y_0 relatively prime to 6, also $y_0^2+3r^2=y^2+3z^2$ for some integers y and z relatively prime to 6. Thus $24n+13=x^2+3y_0^2+9r^2=x^2+3y^2+9z^2$ with x,y,z relatively prime to 6. This proves the universality of $p_5+3p_5+9p_5$ over $\mathbb{Z}$.

(vi) By the Gauss-Legendre theorem (cf. [N96, pp. 17-23]), $8n+3=x^2+y^2+z^2$ for some odd integers x,y,z. Without loss of generality we may assume that $x\not\equiv y \pmod 4$. By Jacobi's identity (2.2), we have $3(8n+3)=u^2+2v^2+6w^2$, where $u=x+y+z$, $v=(x+y)/2-z$ and $w=(x-y)/2$ are odd integers. As u^2+2v^2 is a positive integer divisible by 3, by [S09, Lemma 2.1] we can write $u^2+2v^2=a^2+2b^2$ with a and b relatively prime to 3. Since $a^2+2b^2=u^2+2v^2\equiv 3 \pmod 8$, both a and b are odd. By Lemma 2.2, $b^2+3w^2=c^2+3d^2$ for some integers c and d relatively prime to 6. Thus $24n+9=a^2+2b^2+6w^2=a^2+2c^2+6d^2$ with a,c,d relatively prime to 6. It follows that $p_5+2p_5+6p_5$ is universal over $\mathbb{Z}$.

In view of the above, we have completed the proof of Theorem 1.1. □

3. Proof of Theorem 1.2

Proof of Theorem 1.2. (i) Let $n\in\mathbb{N}$. By part (v) in the proof of Theorem 1.1, there are integers $u,v,w\in\mathbb{Z}$ relatively prime to 6 such that

$$72n+61=24(3n+2)+13=9u^2+3v^2+w^2.$$

Clearly $w^2\equiv 61-3v^2\equiv 7^2 \pmod 9$ and hence $w\equiv\pm 7 \pmod 9$. So there are $x,y,z\in\mathbb{Z}$ such that

$$72n+61=9(2x+1)^2+3(6y-1)^2+(18z-7)^2$$

and hence $n=p_3(x)+p_5(y)+p_{11}(z)$. (Note that $p_{11}(x)=9(x^2-x)/2+x=(9x^2-7x)/2$.)

(ii) Let $n\in\mathbb{N}$. It is easy to see that

$$\begin{aligned}&n=3p_3(x)+p_5(y)+p_7(z)\\ \iff\ &120n+77=5(3(2x+1))^2+5(6y-1)^2+3(10z-3)^2.\end{aligned}$$

Suppose $120n+77=5x^2+5y^2+3z^2$ for some $x,y,z\in\mathbb{Z}$ with z odd. Then $x^2+y^2\equiv 77-3z^2\equiv 2 \pmod 4$ and hence x and y are odd. Note that $3z^2\equiv 77\equiv 12 \pmod 5$ and hence $z\equiv\pm 3 \pmod{10}$. As $5x^2+5y^2\equiv 77\equiv 5 \pmod 3$, exactly one of x and y is divisible by 3. Thus there are $u,v,w\in\mathbb{Z}$ such that

$$120n+77=5(3(2u+1))^2+5(6v-1)^2+3(10w-3)^2.$$

By the above, to prove the universality of $3p_3+p_5+p_7$ over $\mathbb{Z}$, we only need to show that $120n+77=5x^2+5y^2+3z^2$ for some $x,y,z\in\mathbb{Z}$ with z odd.

By (2.3), there are $u,v,w\in\mathbb{Z}$ such that $120n+77=u^2+v^2+3w^2$. As $3w^2\not\equiv 77\equiv 1 \pmod 4$, u or v is odd, say, $2\nmid u$. As $v^2+3w^2\equiv 77-u^2\equiv 4 \pmod 8$, by Lemma 2.1 we may assume that v and w are odd without loss of generality.

We claim that $120n+77=a^2+b^2+3c^2$ for some odd integers a,b,c with $c\equiv\pm2 \pmod 5$. This holds if $w\equiv\pm2 \pmod 5$. Suppose that $w\not\equiv\pm2 \pmod 5$. If $w\equiv\pm1 \pmod 5$, then $u^2+v^2\equiv 77-3w^2\equiv -1 \pmod 5$ and hence u or v is divisible by 5. If $w\equiv 0 \pmod 5$, then $u^2+v^2\equiv 77\equiv 2 \pmod 5$ and hence $u^2\equiv v^2\equiv 1 \pmod 5$. Without loss of generality, we assume that one of v and w is divisible by 5 and the other one is congruent to 1 or -1 modulo 5, we may also suppose that $v\not\equiv w \pmod 4$ (otherwise we may use $-w$ instead of w). By the identity (2.1),

$$v^2+3w^2=\left(\frac{v+3w}{2}\right)^2+3\left(\frac{v-w}{2}\right)^2.$$

(We thank our colleague Hao Pan for suggesting the use of (2.1) in this situation.) Note that both $(v-w)/2$ and $(v+3w)/2=(v-w)/2+2w$ are odd. Also, $(v-w)/2$ is congruent to 2 or -2 modulo 5. This confirms the claim.

By the above, there are odd integers $a,b,c\in\mathbb{Z}$ with $c\equiv\pm2 \pmod 5$ such that $120n+77=a^2+b^2+3c^2$. Since $3c^2\equiv 77 \pmod 5$, we have $5\mid a^2+b^2$ and hence $a^2\equiv(2b)^2 \pmod 5$. Without loss of generality we assume that $a\equiv 2b \pmod 5$. Then $x=(2a+b)/5$ and $y=(a-2b)/5$ are odd integers, and

$$a^2+b^2=(2x+y)^2+(x-2y)^2=5(x^2+y^2).$$

Now we have $120n+77=5(x^2+y)^2+3c^2$ with x,y,c odd. This concludes our proof. $\square$

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